Entropic uncertainty relation for subsequent measurements

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Abstract. Throughout the history, various forms of the uncertainty relationship have been introduced to impose different operational meanings of quantum uncertainty principle. In this work, we will derive new form of uncertainty relationship, using the notion of entropy which quantifies an amount of uncertainty. The entropic uncertainty relationship characterizes the amount of disturbance in an subsequent measurement identifying the relation of projection measurement and its disturbance in the orthogonal basis.

Keywords: Entropic uncertainty relation, Subsequent measurement, Error and disturbance

1 Introduction

Uncertainty principle is at the heart of the quantum mechanics. Through extensive previous investigations, it has been known that various distinctive properties of the quantum mechanics can be derived as a results of the principle. However, many efforts to discover the underlying meaning of the principle have been perpetual due to its diverse feature.

In 1927 [1], Heisenberg proposed the uncertainty relation for the first time,

$$\epsilon(Q)\eta(P) \geq \frac{\hbar}{2}, \tag{1}$$

where $\epsilon(Q)$ is the mean error that occurs when an observer measures the position of a particle, and $\eta(P)$ is the disturbance of the momentum P caused by the position measurement Q and $\hbar$ is the Plank constant. The equation (1) shows that we cannot measure position Q exactly without disturbing momentum P.

The Heisenberg's relation (1) was later developed to arbitrary pair of observables by Robertson [2]. By considering generalized observables A and B, the lower bound is given by commutator of the observables,

$$\sigma(A)\sigma(B) \geq \frac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|, \tag{2}$$

where $\sigma(A)$ is the standard deviation defined as $\sigma(A) = \sqrt{|\langle\psi|(\hat{A}-\langle\hat{A}\rangle)^2|\psi\rangle|}$ and $[\hat{A},\hat{B}]$ represents the commutator, $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. The above relation (2) claims that in an arbitrary state $|\psi\rangle$, a pair of noncommuting observables cannot be well localized simultaneously .

The underlying meaning of two uncertainty relations is not equivalent each other. The difference would be clear as we refer to following three statements of uncertainty relations.[3] : (i) It is impossible to *prepare states* in which position and momentum are simultaneously arbitrarily well localized. (ii) It is impossible to *measure a system's position and momentum* simultaneously . (iii) It is impossible to *measure position without disturbing momentum.* In these statements, position and momentum represent conjugate variables.

According to the statements, we can classify the above relations into different categories. First, the Robertson's relation (2) is equivalent with the statement (i), which denotes a limitation to prepare states in which noncommuting observables are well localized simultaneously. Second, the Heisenber's relation (1) is equivalent with the statement (iii), since they describe a situation that a measurement for position Q cannot avoid the disturbance on P, when we consider two noncommuting observables Q and P.

There were efforts to generalize the Heisenberg's relation (1), but it was not complete until now. Recently, Ozawa derived a universally valid error-disturbance uncertainty relation in 2004 [4] defined as,

$$\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|, \tag{3}$$

where the mean error and the disturbance are defined by $\epsilon(A)^2 = \sum_m ||\hat{M}_m(m-\hat{A})|\psi\rangle||^2$ and $\eta(B)^2 = \sum_m ||[\hat{M}_m,\hat{B}]|\psi\rangle||^2$, respectively, if the apparatus M has a family $\{\hat{M}_m\}$ of measurement operators and $||\ldots||$ denotes the norm of the state vector. This means that the measuring apparatus M has possible outcomes m with probability $p(m) = ||\hat{M}_m|\psi\rangle||^2$ and the state of the object S after the measurement with the outcome m is $\hat{M}_m|\psi\rangle/||\hat{M}_m|\psi\rangle||$. More recently, it was also proved experimentally that the Heisenberg's relation (1) is violated in spin measurements, while the improved relationship (3) is still valid [5].

Inspired by information theoretic interpretation of quantum uncertainty, D. Deustch tried to construct the uncertainty relation in terms of the Shannon entropy in 1983 [6] and it is called the entropic uncertainty relationship (EUR). In 1988, the bound of EUR was improved by Uffink in the following form [7]. When probability distribution is defined as $X = \{p_1, p_2...p_n\}$, $Y = \{q_1, q_2...q_n\}$ and $H(X)$ is Shannon entropy described as $H(X) = -\Sigma_i p_i \log p_i$, the EUR becomes

$$H(X) + H(Y) \geq -2\log c, \quad \text{where} \quad c = |\max_{i,j}\langle x_i|y_j\rangle|, \tag{4}$$

where $\{|x_i\rangle\}$ and $\{|y_j\rangle\}$ are the corresponding complete sets of normalized eigenvectors with respect to operators $\hat{X}$ and $\hat{Y}$, and then p_i and q_j are defined as $|\langle x_i|\psi\rangle|^2$ and $|\langle y_i|\psi\rangle|^2$, respectively. In general, it can be said that en-

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tropy based uncertainty relationship has more fundamental lower bound then the Robertson type of uncertainty relationship, since it is independent of the prepared initial state, unlikely to (2) and (3). However, the EUR in (4) only is limited by the state preparation as like inequality in (2). Thus, in this paper, we generalize EUR in (4) and derive a new bound of EUR reflecting the statement (iii) of [3], error-disturbance consideration.

2 Entropic uncertainty relation for subsequent measurements

In this section, we will show the entropic uncertainty relation obtained in subsequent measurements in oder to reflect the error-disturbance. In subsequent measurements, a probability to obtain an eigenvalue x_i of eigenvector $|x_i\rangle$ after the measurement X is given by $P(x_i) = |\langle\psi|x_i\rangle|^2$. Consequently we obtain the eigenvalue y_j with a probability $P(y_j|x_i) = |\langle x_i|y_j\rangle|^2$ as we measure the outcome state immediately with measurement Y.

Thus, the entropy of the probability distribution for the subsequent measurements is given by,

$$\begin{aligned} H(X,Y) &= -\sum_{i,j} P(x_i)P(y_j|x_i)\log P(x_i)P(y_j|x_i) \\ &= H(X) - \sum_{i,j} P(x_i)P(y_j|x_i)\log P(y_j|x_i), \end{aligned}$$

where $H(X,Y)$ quantifies an amount of uncertainty when a state is measured by measurements X and Y subsequently. As a result, we can obtain a relation,

$$H(X) - \sum_{i,j} P(x_i)P(y_j|x_i)\log P(y_j|x_i) \geq -2\log c, \quad (5)$$

where $c = \max_{i,j} |\langle x_i|y_j\rangle|$ which is independent of the initial state. This relation implies a limitation to measure X without disturbance of Y which has nonorthogonal sets with X. Second term of the left hand side is average entropy caused by the measurement X.

3 Example

Let consider two spin measurement depicted in Fig 1 which was used to verify the relation (3), in the reference [5]. Measurements X, Y are designed to carry out the projective measurements of $\sigma_\phi = \sigma_x \cos\phi + \sigma_y \sin\phi$ and σ_y, respectively. Since each measurement has their own eigenvectors, after the measurements it would be collapsed into spin up state $|+\rangle$ or spin down state $|-\rangle$. In this way, its final result is emerged among 4 possible outcomes described in Fig 1.

The final results appear with the probability distribution which is function of ϕ, and the entropic uncertainty of them is depicted in Fig (2a)$\sim$(2b) in terms of ϕ with various initial states. Fig (2a) shows original entropic uncertainty $H(X) + H(Y)$. On the other hand, Fig (2b) considers the entropic uncertainty relation for the subsequent measurements.

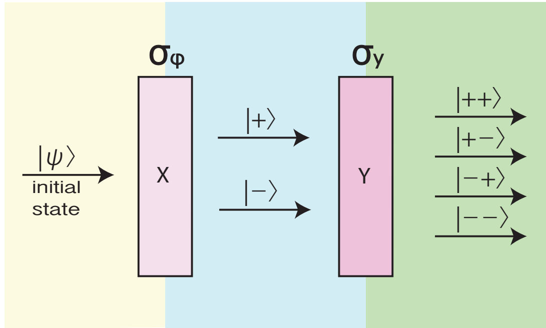
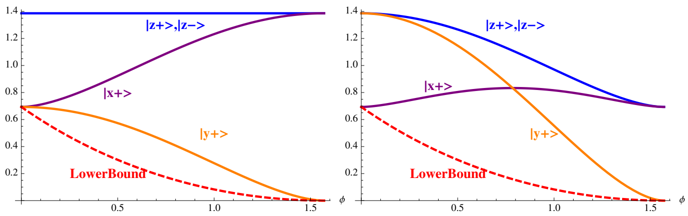


Figure 1: Projection operators X, $\hat{\sigma}_\phi = \hat{\sigma}_x \cos\phi + \hat{\sigma}_y \sin\phi$ and Y, $\hat{\sigma}_y$. The subsequent measurements result in one of four outcomes denoted as $|\pm\pm\rangle, |\pm\mp\rangle$



(a) $H(X) + H(Y)$ (b) $H(X) - \sum_{i,j} P(x_i)P(y_j|x_i)\log P(y_j|x_i)$

Figure 2: Graphs show the left hand side of relations (4), (5) with respect to ϕ. Further, various initial states are considered with different colors.

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